2020/11/25.

G3

TD3. TD4 TD5: ~~TD6~~

prochain DS — mais finale

TD5.

EX5: ① $Y(0) = 10 \times 1.05^0 = 10$.

$Y(0.5) = 10 \times 1.05^{0.5} = 10 \times \sqrt{1.05} \simeq 10.247$

$Y(2) = 10 \times 1.05^2 = 11.025$

② le taux de croissance du PIB: (méthode 1°)

$$\boxed{\gamma = \frac{Y'(t)}{Y(t)}}$$

posons $f(t) = \ln(Y(t))$. on a $\boxed{f'(t) = \gamma}$

$\Rightarrow f(t) = \ln(10 \times 1.05^t) = \ln 10 + \ln(1.05)^t$
$= \ln 10 + t \ln(1.05)$

On sait que $(\ln t)' = \frac{1}{t}$

$$(\ln Y(t))' = \frac{1}{Y(t)} \cdot Y'(t) = \frac{Y'(t)}{Y(t)} = \gamma.$$

$\Rightarrow f(t) = \underbrace{\ln 10}_{A} + t \underbrace{\ln(1,05)}_{B} = A + tB$

$\Rightarrow \underbrace{f'(t)}_{=\gamma} = \ln(1,05)$

Donc le taux de croissance du PIB est $\gamma = \ln(1.05)$
$\simeq 0.04879$
$= 4.879\%$

méthode 2°.

le taux de croissance : $\gamma = \dfrac{Y'(t)}{Y(t)}$

$Y(t) = 10 \times \underline{1.05^{t}}$.

$\boxed{(a^x)' = a^x \ln a}$

a est Constant

$Y'(t) = 10 \times (1.05^{t}) \ln(1.05)$

$\Rightarrow \gamma = \dfrac{Y'(t)}{Y(t)} = \dfrac{\cancel{10} \times \cancel{1.05^{t}} \times \ln(1.05)}{\cancel{10} \times \cancel{1.05^{t}}}$

$= \ln(1.05) \approx 4.879\%$

③ le prix double pour t tq. $Y(t) = 2\,Y(0)$

$10 \times 1.05^{t} = 2 \times Y(0)$

$\Rightarrow \cancel{10} \times 1.05^{t} = 2 \times \cancel{10}$

$\Rightarrow \boxed{1.05^{t} = 2}$

$\Rightarrow \ln 1.05^{t} = \ln 2$

$\ln(a)^t = t \ln a$

$\Rightarrow t \ln(1.05) = \ln 2$

$\Rightarrow t = \dfrac{\ln 2}{\ln(1.05)} \approx 14.2$ ans

④ même pour $Z(t) = 20 \times 1.02^{t}$. $\gamma_Z = 1.98\%$

⑤ $Y(t) = Z(t)$

c-à-d : $10 \times 1.05^{t} = 20 \times 1.02^{t}$

$$\Rightarrow \quad 1.05^t = 2 \times 1.02^t$$

$$\Rightarrow \quad \boxed{\left(\frac{1.05}{1.02}\right)^t = 2}$$

(voir Ex3. ①.
Ex5. ②)

$$\Rightarrow \ln\left(\frac{1.05}{1.02}\right)^t = \ln 2$$

$$\Rightarrow \quad t \ln\left(\frac{1.05}{1.02}\right) = \ln 2$$

$$\Rightarrow \quad t = \frac{\ln 2}{\frac{\ln(1.05)}{\ln(1.02)}} \simeq 23{,}9 \text{ ans}$$

Ex6: $A_u(x) = -\frac{u''(x)}{u'(x)}$ $\qquad R_u(x) = -x \cdot \frac{u''(x)}{u'(x)}$

① $u(x) = ax + b. \Rightarrow u'(x) = a, \quad u''(x) = 0.$

$\Rightarrow \quad A_u(x) = 0 \quad . \quad R_u(0) = 0$

② $u(x) = \ln x.$ pour toute $x > 0$

$\Rightarrow u'(x) = \frac{1}{x}. \qquad u''(x) = -1 \times \frac{1}{x^2} = -\frac{1}{x^2}$

$$\Rightarrow A_u(x) = -\frac{u''(x)}{u'(x)} = -\frac{-\frac{1}{x^2}}{\frac{1}{x}} = \frac{1}{x^2} \cdot \frac{x}{1} = \frac{1}{x}$$

$$R_u(x) = -x \frac{u''(x)}{u'(x)} = x A_u(x) = x \cdot \frac{1}{x} = 1$$

③ $u(x) = \frac{1}{1-r} x^{1-r}.$

$u'(x) = \frac{1}{\cancel{1-r}} \times (\cancel{1-r}) \times x^{-r} = x^{-r} \quad . \quad u''(x) = -r x^{-r-1}$

$$\Rightarrow A_u(x) = -\frac{u''(x)}{u'(x)} = -\frac{-r x^{-r-1}}{x^{-r}} = r x^{-1} = \frac{r}{x}$$

$$R_u(x) = x A_u(x) = x \cdot \frac{r}{x} = r$$

④ $u(x) = -e^{-ax} \quad \Rightarrow \quad u'(x) = -e^{-ax} \times (-a) = a e^{-ax}$

$\Rightarrow \quad u''(x) = a e^{-ax} \times (-a) = -a^2 e^{-ax}$

$\Rightarrow A_u(x) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2 e^{-ax}}{a e^{-ax}} = a. \qquad R_u(x) = x A_u(x) = ax$

TD6. Taylor-Young : $f(x_0+x) = f(x_0) + f'(x_0)x + \frac{1}{2!}f''(x_0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)x^n + x^n \varepsilon(x)$

Ex 2 : ① ②   Ex 3 : ① ②

Ex 2 : À l'aide de la formule de T-Y, calculer un développement limité d'ordre 3 au voisinage 0 de :

① $f(x) = \sqrt{1+x}$

② $g(x) = \ln(1+x)$

② $x_0 = 0$.

$$g(x_0+x) = g(0+x) = g(0) + g'(0)x + \frac{1}{2!}g''(0)x^2 + \frac{1}{3!}g'''(0)x^3 + x^3\varepsilon(x)$$

$g(0) = \ln(1+0) = \ln 1 = 0$

$g'(x) = \frac{1}{1+x} \Rightarrow g'(0) = \frac{1}{1+0} = 1$

$g''(x) = -\frac{1}{(1+x)^2} = -(1+x)^{-2} \Rightarrow g''(0) = -\frac{1}{(1+0)^2} = -1$

$g'''(x) = (-1)\times(-2)\times(1+x)^{-3} = 2(1+x)^{-3} \Rightarrow g'''(0) = 2\times(1+0)^{-3} = 2$

Donc $g(0+x) = 0 + 1\times x + \frac{1}{2}\times(-1)x^2 + \frac{1}{3\times 2}\times 2\times x^3 + x^3\varepsilon(x)$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3\varepsilon(x)$$

DL d'ordre 3 au voisinage 0.

Ex 3. À l'aide de la formule de T-Y, calculer DL d'ordre 2 de

① $f(x) = \sqrt{1+x+x^2}$ au voisinage de $x_0 = 0$

② $g(x) = \ln(2+2x+x^2)$, au voisinage de $x_0 = 2$.

② $g(x_0+x) = g(2+x) = g(2) + g'(2)x + \frac{1}{2!}g''(2)x^2 + x^2\varepsilon(x)$

$g(2) = \ln(2+2\times 2+2^2) = \ln(10)$

$g(x) = \ln(2+2x+x^2)$, on pose $u = 2+2x+x^2$

$u' = 2+2x$

$\Rightarrow g'(x) = \frac{1}{u}\cdot u' = \frac{1}{2+2x+x^2}\cdot(2+2x)$

$$\Rightarrow g'(2) = \frac{2+2\times 2}{2+2\times 2+2^2} = \frac{6}{10} = \frac{3}{5}$$

$g'(x) = \dfrac{2+2x}{2+2x+x^2}$. On pose $u = 2+2x+x^2$.

$v = 2+2x$

$\Rightarrow u' = 2+2x, \quad v' = 2$

$$\Rightarrow g''(x) = \frac{uv' - u'v}{u^2} = \frac{(2+2x+x^2)\times 2 - (2+2x)\times(2+2x)}{(2+2x+x^2)^2}$$

$$\Rightarrow g''(2) = \frac{(2+2\times 2+2^2)\times 2 - (2+2\times 2)\times(2+2\times 2)}{(2+2\times 2+2^2)^2}$$

$$= \frac{10\times 2 - 6\times 6}{10^2} = -\frac{16}{100} = -\frac{4}{25}$$

$$\Rightarrow g(2+x) = \ln(10) + \frac{3}{5}x + \frac{1}{2}\times\left(-\frac{4}{25}\right)x^2 + x^2\varepsilon(x)$$

$$= \ln 10 + \frac{3}{5}x - \frac{2}{25}x^2 + x^2\varepsilon(x)$$

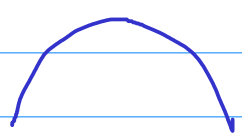
Ex4: Concavité & Convexité.

① $f(x) = 7x^4 + 8x^2$

② $g(x) = 7x^4 - 8x^2$

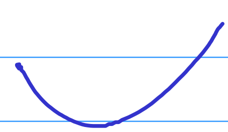
③ $h(x) = 2\ln x - 4x^3$.

concave



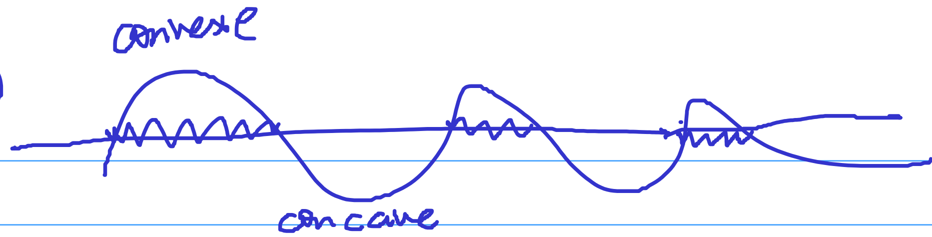
$f''(x) \leq 0$

convexe



$f''(x) \geq 0$

$f''(x)$



① $f(x) = 7x^4 + 8x^2$

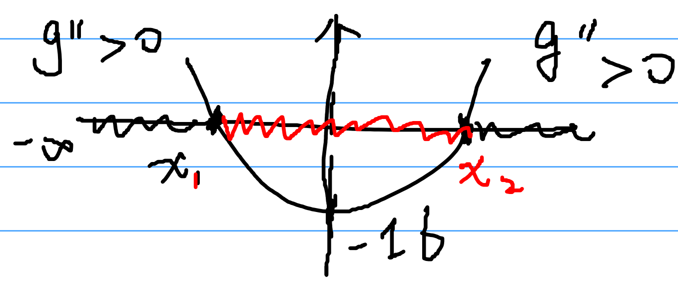
$f'(x) = 28x^3 + 16x$, $f''(x) = \underbrace{84x^2 + 16}_{>0}$

f est strictement convexe sur $\mathbb{R}$.

② $g(x) = 7x^4 - 8x^2$

$g'(x) = 28x^3 - 16x$, $\underline{g''(x) = 84x^2 - 16.}$

$g(x)$ n'est pas concave sur $\mathbb{R}$.
n'est pas convexe sur $\mathbb{R}$.



mais
$g(x)$ est convexe sur

$]-\infty, -\sqrt{\frac{4}{21}}]$, $[\sqrt{\frac{4}{21}}, +\infty[$

$84x^2 - 16 = 0$

$g(x)$ est concave sur

$\Rightarrow x^2 = \frac{16}{84} = \frac{4}{21}$

$[-\sqrt{\frac{4}{21}}, \sqrt{\frac{4}{21}}]$.

③ $h(x) = 2\ln x - 4x^3$

$h'(x) = 2\frac{1}{x} - 12x^2$, $h''(x) = -2\frac{1}{x^2} - 24x$

Taylor-Young : $f(x_0+x) = f(x_0) + f'(x_0)x + \frac{1}{2!}f''(x_0)x^2 + \dots$

$\frac{1}{n!}f^{(n)}(x_0)x^n + x^n\varepsilon(x)$

TD6.

Ex2. ① ② Ex3. ① . ②

$n! = 1\times2\times3\times\dots\times n$

① $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$ $\boxed{\text{d'ordre } 3}$.

$$f(0+x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + x^3\varepsilon(x)$$

$f(0) = \sqrt{1+0} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \Rightarrow f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2}\times(-\frac{1}{2})\times(1+x)^{-\frac{3}{2}} = -\frac{1}{4}(1+x)^{-\frac{3}{2}} \Rightarrow f''(0) = -\frac{1}{4}$

$f'''(x) = -\frac{1}{4}\times(-\frac{3}{2})(1+x)^{-\frac{5}{2}} = \frac{3}{8}(1+x)^{-5/2} \Rightarrow f'''(0) = \frac{3}{8}$

Donc. $f(0+x) = f(x) = \sqrt{1+x}$

$= 1 + \frac{1}{2}x + \frac{1}{2!}\times(-\frac{1}{4})x^2 + \frac{1}{3!}\times\frac{3}{8}x^3 + x^3\varepsilon(x)$

3×2

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + x^3\varepsilon(x)$

Ex3 : ① $f(x) = \sqrt{1+x+x^2} = (1+x+x^2)^{1/2}$

par T-Y : $f(0+x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + x^2\varepsilon(x)$

$f(0) = \sqrt{1+0+0} = 1$

$f'(x) = \underbrace{\frac{1}{2}(1+x+x^2)^{-\frac{1}{2}}}_{u}\cdot\underbrace{(1+2x)}_{v} \Rightarrow f'(0) = \frac{1}{2}(1+0+0)^{-\frac{1}{2}}(1+0) = \frac{1}{2}$

$u = (1+x+x^2)^{-\frac{1}{2}}$ $v = 1+2x$

$u' = -\frac{1}{2}(1+x+x^2)^{-\frac{1}{2}}(1+2x)$ $(uv)' = u'v + uv'$

$v' = 2$

Donc $f''(x) = \frac{1}{2}(u'v + uv')$

$= \frac{1}{3}\left(-\frac{1}{2}(1+x+x^2)^{-\frac{3}{2}}(1+2x)(1+2x) + (1+x+x^2)^{-\frac{1}{2}}\times 2\right)$

$= -\frac{1}{4}(1+x+x^2)^{-\frac{3}{2}}(1+2x)^2 + (1+x+x^2)^{-\frac{1}{2}}$

$\Rightarrow f''(0) = -\frac{1}{4} + 1 = \frac{3}{4}$

$\Rightarrow f(0+x) = f(0) + \underline{f'(0)}\, x + \frac{1}{2!} f''(0)\, x^2 + x^2 \varepsilon(x)$

$= 1 + \frac{1}{2} x + \frac{1}{2} \times \frac{3}{4} x^2 + x^2 \varepsilon(x)$

$= 1 + \frac{1}{2} x + \frac{3}{8} x^2 + x^2 \varepsilon(x)$